QN.→ If $u = \log(x^3+y^3+z^3-3xyz)$ show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$

Ans.→ : we know that

$$x^3+y^3+z^3-3xyz$$
$$=(x+y+z)(x+y\omega+z\omega^2)$$
$$=(x+y\omega^2+z\omega)$$

where ω and ω^2 we the imaginary cube root of unity

and $1+\omega+\omega^2=0$

and $\omega^3=1$

$$u=\log(x+y+z)(x+y\omega+z\omega^2)(x+y\omega^2+z\omega)$$
$$=\log(x+y+z)+\log(x+y\omega+z\omega^2)+\log(x+y\omega^2+z\omega)$$

$$u=\log(x+y+z)+\log(x+\omega y+z\omega^2)+\log(x+\omega^2 y+z\omega)$$

$$\frac{\partial u}{\partial x}=\frac{1}{x+y+z}+\frac{1}{x+\omega y+z\omega^2}+\frac{1}{x+\omega^2 y+z\omega} \quad \text{—(1)}$$

$$\frac{\partial u}{\partial y}=\frac{1}{x+y+z}+\frac{\omega}{x+\omega y+z\omega^2}+\frac{\omega^2}{x+\omega^2 y+z\omega} \quad \text{—(2)}$$

$$\frac{\partial u}{\partial z}=\frac{1}{x+y+z}+\frac{\omega^2}{x+\omega y+z\omega^2}+\frac{\omega}{x+\omega^2 y+z\omega} \quad \text{—(3)}$$

(1) + (2) + (3)

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{x+y+z}+(1+\omega+\omega^2)\frac{1}{x+\omega y+z\omega^2}+$$

$$(1+\omega^2+\omega)\frac{1}{x+\omega^2 y+z\omega}$$

$$=\frac{3}{x+y+z}=v$$

$$v=\frac{3}{x+y+z}=3(x+y+z)^{-1}$$

$$\frac{\partial v}{\partial x}=-3(x+y+z)^{-2}\times 1=\frac{-3}{(x+y+z)^2}$$

sim. $\frac{\partial v}{\partial y}=\frac{-3}{(x+y+z)^2}$

$$\frac{\partial v}{\partial z}=\frac{-3}{(x+y+z)^2}$$

$$\therefore \frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}=\frac{-9}{(x+y+z)^2}$$

$$L.H.S.=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)v$$

$$=\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}=\frac{-9}{(x+y+z)^2}$$ Proved.

(23) Q.N.→ If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x+y+z)^2}$$

Ans.→ ∵ we know that

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

$$u = \log(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

$$= \log(x+y+z) + \log(x^2+y^2+z^2-xy-yz-zx)$$

Diff. partially w.r.t. x keeping y and z as constants.

$$\frac{\partial u}{\partial x} = \frac{1}{x+y+z} + \frac{2x-y-z}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x} = -\frac{1}{(x+y+z)^2} + \frac{2}{(x^2+y^2+z^2+xy-yz-zx)} - \frac{(2x-y-z)(2x-y-z)}{(x^2+y^2+z^2-xy-yz-zx^2)}$$

$$= -\frac{1}{(x+y+z)^2} + \frac{-(4x^2+y^2+z^2-4xy+2yz-4xz)}{(x^2+y^2+z^2-xy-yz-zx)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{(x+y+z)^2} + \frac{y^2+z^2+2xy+2xz-2x^2-4yz}{(x^2+y^2+z^2-xy-yz-zx)^2}$$

Similarly $$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{(x+y+z)^2} + \frac{z^2+x^2+2yz+2yx-2y^2-4zx}{(x^2+y^2+z^2-xy-yz-zx)^2}$$

$$\frac{\partial^2 u}{\partial z^2} = -\frac{1}{(x+y+z)^2} + \frac{x^2+y^2+2zx+2zy-2z^2-4xy}{(x^2+y^2+z^2-xy-yz-zx)^2}$$

Hence, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{3}{(x+y+z)^2}$

(25) If $u=\cos^{-1}\frac{x-y}{x+y}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$.

Ans. → we have, $u=\cos^{-1}\frac{x-y}{x+y}$

$\therefore \cos u=\frac{x-y}{x+y}=\frac{\cancel{x}(1-\frac{y}{x})}{\cancel{x}(1+\frac{y}{x})}=0 \times \cancel{\cos u}=0$ (1)

Hence v is a homogeneous function of x and y of degre zero.

Hence, from Euller's theorm,

$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=0\times v=0$

Now, $v=\cos u$ — (2)

$\therefore$ Diff: (2) partially w.r.t. to x, we have

$\frac{\partial v}{\partial x}=-\sin u\frac{\partial u}{\partial x}$

$x\frac{\partial v}{\partial x}=-x\sin u\frac{\partial u}{\partial x}$ — (a)

Simil $y\frac{\partial v}{\partial y}=-y\sin u\frac{\partial u}{\partial y}$ — (b)

(a) + (b)

$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=-\sin u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$

$0=-\sin u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$

or, $\frac{0}{\sin u} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ or, $0 = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

(26) If $u = \sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ Show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Ans.→ we, have, $u = \sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

$\therefore \sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \frac{\sqrt{x}\left(1-\frac{\sqrt{y}}{\sqrt{x}}\right)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)} = 0 \times v$ — (1)

Hence v is a homogeneous function ~~from~~ of x and y of degree 0,

Hence, from Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0 \times v = 0$$

Now, $\sin u = v$ — (2)

$\therefore$ Diff (2) partially w.r.t. x we, have,

$$\frac{\partial v}{\partial x} = \cos u\frac{\partial u}{\partial x}$$

$$x\frac{\partial v}{\partial x} = x\cos u\frac{\partial u}{\partial x} \text{ — (a)}$$

Sm. $y\frac{\partial v}{\partial y} = y\cos u\frac{\partial u}{\partial y}$ — (b)

(a) + (b)

$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$ or, $\frac{0}{\cos u} = \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$

or, $0 = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$